1 Problem Statement

Obtain the angular dependence of the rate for the emission to a single photon (momentum $\mathbf{k}$ and circular polarization λ) for the electric quadrapole transition from $\ell = 2$ to $\ell = 0$. Neglect electron spin.

2 Calculation

$$H' = \frac{1}{2m}\left(-e\mathbf{p}\cdot\mathbf{A} - e\mathbf{A}\cdot\mathbf{p} + e^2A^2\right)$$

We are after a single photon decay rate, and the A^2 term increases the photon number by 0 or ± 2 so we can ignore this term. Also, in the radiation (Coulomb) gage, we can change the order of $\mathbf{p}$ and $\mathbf{A}$, as $\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)\cdot\left(\nabla e^{-i\mathbf{k}\cdot\mathbf{r}}\right) = 0$. Additionally we are only interested in a final state with one additional photon.

$$\langle n_f\,\ell_f\,m_f;\gamma(\mathbf{k},\,\lambda)\mid H'\mid n_i\,\ell_i\,m_i;0\rangle$$

$$= \langle n_f\,\ell_f\,m_f;\gamma(\mathbf{k},\,\lambda)\mid\left(-\frac{e}{m}\mathbf{A}\cdot\mathbf{p}\right)\mid n_i\,\ell_i\,m_i;0\rangle$$

$$= \langle n_f\,\ell_f\,m_f\mid\left(-\frac{e}{m}e^{-i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)\cdot\mathbf{p}\right)\mid n_i\,\ell_i\,m_i\rangle e^{i\omega t}$$

Next we use $e^{-i\mathbf{k}\cdot\mathbf{r}} \approx 1 - i\mathbf{k}\cdot\mathbf{r}$. The matrix element from the 1 is zero for a quadrapole transition, leaving

$$\langle n_f\,0\,0\mid\left(i\frac{e}{m}\mathbf{k}\cdot\mathbf{r}\,\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)\cdot\mathbf{p}\right)\mid n_i\,2\,m_i\rangle e^{i\omega t}$$

This matrix element includes the quadrapole electric interaction as well as the interaction with the magnetic dipole due to orbital angular momentum.

$$\begin{aligned}&\mathbf{k}\cdot\mathbf{r}\,\boldsymbol{\epsilon}^*(\mathbf{k},\lambda)\cdot\mathbf{p}\\ &= \frac{1}{2}\left[\mathbf{k}\cdot\mathbf{r}\,\boldsymbol{\epsilon}^*\cdot\mathbf{p} + \boldsymbol{\epsilon}^*\cdot\mathbf{r}\,\mathbf{k}\cdot\mathbf{p}\right]\\ &+ \frac{1}{2}\left[\mathbf{k}\cdot\mathbf{r}\,\boldsymbol{\epsilon}^*\cdot\mathbf{p} - \boldsymbol{\epsilon}^*\cdot\mathbf{r}\,\mathbf{k}\cdot\mathbf{p}\right]\end{aligned}$$

Orbital angular momentum of the electron should have an associated magnetic dipole moment, resulting in an energy proportional to $\mathbf{B}\cdot\mathbf{L} = (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\mathbf{r}\times\mathbf{p})$. For our calculation, $\boldsymbol{\nabla}\times\mathbf{A} \to -i\mathbf{k}\times\boldsymbol{\epsilon}^*$, keeping 0-th order of $e^{-i\mathbf{k}\cdot\mathbf{r}}$ after taking the curl.

$$(\mathbf{k}\times\boldsymbol{\epsilon}^*)\cdot(\mathbf{r}\times\mathbf{p}) =$$

$$\epsilon_{iab}\epsilon_{imn}k_a\epsilon_b^* r_m p_n$$

$$= (\delta_{am}\delta_{bn} - \delta_{an}\delta_{bm})\,k_a\epsilon_b^* r_m p_n$$

$$= \mathbf{k}\cdot\mathbf{r}\,\boldsymbol{\epsilon}^*\cdot\mathbf{p} - \boldsymbol{\epsilon}^*\cdot\mathbf{r}\,\mathbf{k}\cdot\mathbf{p}$$

We will not consider these terms for the remainder of this calculation, as they correspond to a magnetic interaction, and it can be verified the above has $\Delta l = 0$.

$$\boldsymbol{\epsilon}^* \cdot \mathbf{p}\, \mathbf{k} \cdot \mathbf{r} + \mathbf{k} \cdot \mathbf{p}\, \boldsymbol{\epsilon}^* \cdot \mathbf{r}$$

$$\begin{aligned}
&= \sum_{m,m'=0,\pm 1} (-1)^{m+m'} \left(\epsilon^*_m k_{m'} + \epsilon^*_{m'} k_m\right) p_{-m} r_{-m'} \\
&= 2\epsilon^*_+ k_+ \underbrace{p_- r_-}_{Q_{2,-2}} + 2\epsilon^*_- k_- \underbrace{p_+ r_+}_{Q_{2,+2}} \\
&\quad - \left(\epsilon^*_0 k_+ + \epsilon^*_+ k_0\right) \underbrace{\left(p_0 r_- + p_- r_0\right)}_{\sqrt{2} Q_{2,-1}} \\
&\quad - \left(\epsilon^*_0 k_- + \epsilon^*_- k_0\right) \underbrace{\left(p_0 r_+ + p_+ r_0\right)}_{\sqrt{2} Q_{2,+1}} \\
&\quad + \left(\epsilon^*_+ k_- + \epsilon^*_- k_+\right) \left(p_- r_+ + p_+ r_-\right) \\
&\quad + 2\epsilon^*_0 k_0 p_0 r_0
\end{aligned}$$

by orthogonality

$$0 = \boldsymbol{\epsilon}^* \cdot \mathbf{k} = \epsilon^*_0 k_0 - \epsilon^*_- k_+ - \epsilon^*_+ k_-$$

$$\epsilon^*_0 k_0 = \epsilon^*_- k_+ + \epsilon^*_+ k_-$$

thus

$$\boldsymbol{\epsilon}^* \cdot \mathbf{p}\, \mathbf{k} \cdot \mathbf{r} + \mathbf{k} \cdot \mathbf{p}\, \boldsymbol{\epsilon}^* \cdot \mathbf{r} =$$

$$2\epsilon^*_+ k_+ Q_{2,-2} + 2\epsilon^*_- k_- Q_{2,+2}$$

$$-\sqrt{2}\left(\epsilon^*_0 k_+ + \epsilon^*_+ k_0\right) Q_{2,-1} - \sqrt{2}\left(\epsilon^*_0 k_- + \epsilon^*_- k_0\right) Q_{2,+1}$$

$$+\sqrt{6}\left(\epsilon^*_- k_+ + \epsilon^*_+ k_-\right) Q_{2,0}$$

Now, we may attempt to express the coefficients of $Q_{2,m}$ in spherical coordinates of $\mathbf{k}$,and relate the coefficients to the small Wigner d-matrix elements.

$$\hat{\boldsymbol{\phi}} = (-\sin\phi,\, \cos\phi,\, 0)$$

$$\hat{\boldsymbol{\theta}} = (\cos\phi\cos\theta,\, \cos\phi\sin\theta,\, -\sin\theta)$$

expressing the circular polarization vectors as a function of the angular coordinates of the emitted photon.

$$\boldsymbol{\epsilon}\left(\mathbf{k}, \pm\right) = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos\phi\cos\theta \mp i\sin\theta,\, \cos\phi\sin\theta \pm i\cos\theta,\, -\sin\theta\right)$$

$$\epsilon_+ = \frac{1}{\sqrt{2}}\left(-\epsilon_x - i\epsilon_y\right)$$

$$= \frac{1}{2}\left(-\cos\phi\cos\theta \pm i\sin\theta - i\cos\phi\sin\theta \pm \cos\theta\right)$$

$$= \frac{1}{2}\left(-\cos\theta e^{i\phi} \pm e^{i\phi}\right) = \frac{1}{2}\left(\pm 1 - \cos\theta\right)e^{i\phi}$$

$$\epsilon_- = \frac{1}{\sqrt{2}}\left(\epsilon_x - i\epsilon_y\right)$$

$$= \frac{1}{2}\left(\cos\phi\cos\theta \mp i\sin\theta - i\cos\phi\sin\theta \pm \cos\theta\right)$$

$$= \frac{1}{2}\left(\cos\theta e^{-i\phi} \pm e^{-i\phi}\right) = \frac{1}{2}\left(\pm 1 + \cos\theta\right)e^{-i\phi}$$

To summarize the components of vectors in the spherical basis denoted by the $-, 0$, or $+$ subscript are as follows.

$$\epsilon_+ = \frac{1}{2}\left(\pm 1 - \cos\theta\right)e^{i\phi} \qquad \epsilon_- = \frac{1}{2}\left(\pm 1 + \cos\theta\right)e^{-i\phi} \qquad \epsilon_0 = -\frac{1}{\sqrt{2}}\sin\theta$$

$$\epsilon_+^* = \frac{1}{2}\left(\pm 1 - \cos\theta\right)e^{-i\phi} \qquad \epsilon_-^* = \frac{1}{2}\left(\pm 1 + \cos\theta\right)e^{i\phi} \qquad \epsilon_0^* = -\frac{1}{\sqrt{2}}\sin\theta$$

$$\mathbf{k} = k\left(\cos\phi\sin\theta,\, \sin\phi\sin\theta,\, \cos\theta\right)$$

$$k_+ = k\frac{1}{\sqrt{2}}\left(-\cos\theta\sin\phi - i\sin\phi\sin\theta\right) = -k\frac{\sin\theta\, e^{i\phi}}{\sqrt{2}}$$

$$k_- = k\frac{1}{\sqrt{2}}\left(\cos\theta\sin\phi - i\sin\phi\sin\theta\right) = k\frac{\sin\theta\, e^{-i\phi}}{\sqrt{2}}$$

$$k_0 = \cos\theta$$

Now we use the above to express the coefficients of the spherical tensor operators.

$$\epsilon_+^* k_+ = \mp \frac{k}{2\sqrt{2}} (1 \pm \cos\theta) \sin\theta = \pm \frac{k}{\sqrt{2}} d^{(2)}_{2,\pm 1}(\theta)$$

$$\epsilon_-^* k_- = \pm \frac{k}{2\sqrt{2}} (1 \pm \cos\theta) \sin\theta = \mp \frac{k}{\sqrt{2}} d^{(2)}_{2,\pm 1}(\theta)$$

$$\begin{aligned}\epsilon_-^* k_+ + \epsilon_+^* k_- &= \mp \frac{k}{2\sqrt{2}} (1 \pm \cos\theta) \sin\theta e^{i2\phi} \pm \frac{k}{2\sqrt{2}} (1 \mp \cos\theta) \sin\theta e^{-i2\phi} \\ &= \frac{k}{\sqrt{2}} \left(\pm d^{(2)}_{2,\pm 1}(\theta) e^{i2\phi} \mp d^{(2)}_{2,\mp 1}(\theta) e^{-i2\phi} \right)\end{aligned}$$

The coefficients of $Q_{2,\pm 1}$ do not seem to be easily expressed in terms of single d-matrix elements.

$$\epsilon_0^* k_+ + \epsilon_+^* k_0 = \frac{1}{2} \sin^2\theta e^{i\phi} + \frac{1}{2} (\pm 1 - \cos\theta) \cos\theta e^{-i\phi}$$

$$\epsilon_0^* k_- + \epsilon_-^* k_0 = -\frac{1}{2} \sin^2\theta e^{i\phi} + \frac{1}{2} (\pm 1 + \cos\theta) \cos\theta e^{i\phi}$$

However we can, at least, express them with the $\ell = 1$ d-matrix elements.

$$\begin{aligned}\epsilon_+^* &= \pm \frac{1}{2} (1 \mp \cos\theta) e^{-i\phi} &&= \pm d^{(1)}_{1,\mp 1}(\theta) e^{-i\phi} \\ \epsilon_-^* &= \pm \frac{1}{2} (1 \pm \cos\theta) e^{i\phi} &&= \pm d^{(1)}_{1,\pm 1}(\theta) e^{i\phi} \\ \epsilon_0^* &= -\frac{1}{\sqrt{2}} \sin\theta &&= d^{(1)}_{1,0}(\theta)\end{aligned}$$

$$k_+ = k d^{(1)}_{1,0}(\theta) e^{i\phi} \qquad k_- = -k d^{(1)}_{1,0}(\theta) e^{-i\phi} \qquad k_0 = d^{(1)}_{0,0}(\theta)$$

$$\epsilon_0^* k_+ + \epsilon_+^* k_0 = k d^{(1)}_{1,0}(\theta) d^{(1)}_{1,0}(\theta) e^{i\phi} \pm k d^{(1)}_{1,\mp 1}(\theta) d^{(1)}_{0,0}(\theta) e^{-i\phi}$$

$$\epsilon_0^* k_- + \epsilon_-^* k_0 = -k d^{(1)}_{1,0}(\theta) d^{(1)}_{1,0}(\theta) e^{-i\phi} \pm k d^{(1)}_{1,\pm 1}(\theta) d^{(1)}_{0,0}(\theta) e^{i\phi}$$

Using the above one may arrive at this final result.

$$\boldsymbol{\epsilon}^* \cdot \mathbf{p}\, \mathbf{k} \cdot \mathbf{r} + \mathbf{k} \cdot \mathbf{p}\, \boldsymbol{\epsilon}^* \cdot \mathbf{r} =$$

$$\pm\sqrt{2} k d^{(2)}_{2,\pm 1}(\theta) Q_{2,-2} + \mp\sqrt{2} k d^{(2)}_{2,\pm 1}(\theta) Q_{2,+2}$$

$$-\sqrt{2} k \left(d^{(1)}_{1,0}(\theta) d^{(1)}_{1,0}(\theta) e^{i\phi} \pm d^{(1)}_{1,\mp 1}(\theta) d^{(1)}_{0,0}(\theta) e^{-i\phi} \right) Q_{2,-1}$$

$$-\sqrt{2} k \left(-d^{(1)}_{1,0}(\theta) d^{(1)}_{1,0}(\theta) e^{-i\phi} \pm d^{(1)}_{1,\pm 1}(\theta) d^{(1)}_{0,0}(\theta) e^{i\phi} \right) Q_{2,+1}$$

$$+k\sqrt{3} \left(\pm d^{(2)}_{2,\pm 1}(\theta) e^{i2\phi} \mp d^{(2)}_{2,\mp 1}(\theta) e^{-i2\phi} \right) Q_{2,0}$$