1. Let $M = \overline{\mathbb{B}}^n$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point. Show how to give it a smooth structure such that every smooth interior chart is a smooth chart for the standard smooth structure on $\mathbb{B}^n$. [Consider the map $\pi \circ \sigma^{-1}: \mathbb{R}^n \to \mathbb{R}^n$, where $\sigma: \mathbb{S}^n \to \mathbb{R}^n$ is the stereographic projection (problem 1.7) and π is a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ that omits some coordinate other than the last.]

Solution:

The Hausdorff and countable basis condition are immediate as $\overline{\mathbb{B}}^n$ is a subspace of $\mathbb{R}^n$.

Since the identity map $i: \mathbb{B}^n \to \mathbb{B}^n$ is homeomorphism, every interior point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Thus each point in $\mathbb{B}^n$ is interior point and i is smooth interior chart.

Let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection that omits the i^{th} coordinate ($i \neq n+1$), that is,

$\pi(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1})$, and a let

$\sigma^{-1}: \mathbb{R}^n \to \mathbb{S}^n$, $\sigma^{-1}(x^1, \dots, x^n) = \frac{(2x^1, \dots, 2x^n, |x|^2 - 1)}{|x|^2+1}$, where $|x|^2 = \sum_{i=1}^n (x^i)^2$.

For each unit ball in $\mathbb{R}^n$ consider the map $\varphi = \pi \circ \sigma^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.

Then $\varphi(x) = \pi \circ \sigma^{-1}(x^1, \dots, x^n)$

$$= \pi\left(\frac{(2x^1, \dots, 2x^n, |x|^2 - 1)}{|x|^2+1}\right)$$

$$= \frac{1}{|x|^2+1}(2x^1, \dots, 2x^{i-1}, 2x^{i+1}, \dots, |x|^2 - 1).$$

Now since $|x|^2 - 1 \leq 0$ for $x \in \overline{\mathbb{B}}^n$, we have

$\varphi\left(\overline{\mathbb{B}}^n\right) = \left\{\varphi(x): x \in \overline{\mathbb{B}}^n\right\} = \mathbb{R}^{n-1} \times (-\infty, 0]$ =the lower half space but one can send it to the upper half space by reflection.

Clearly φ is continuous, and it is open map as it is a composition of open maps.

Let U^+ be a neighborhood of a point $p \in \mathbb{S}^n$, such that $U^+ \subset \left\{x \in \overline{\mathbb{B}}^n : x^i \geq 0\right\}$.

Then to show that φ is one to one on U^+; let $\varphi(x) = \varphi(y)$.

Then $\frac{1}{|x|^2+1}\left(2x^1, \dots, 2x^{i-1}, 2x^{i+1}, \dots,, |x|^2 - 1\right) = \frac{1}{|y|^2+1}(2y^1, \dots, 2y^{i-1}, 2y^{i+1}, \dots,, |y|^2 - 1)$

$$\Longrightarrow \frac{2x^1}{|x|^2+1} = \frac{2y^1}{|y|^2+1}, \dots, \frac{2x^{i-1}}{|x|^2+1} = \frac{2y^{i-1}}{|y|^2+1}, \frac{2x^{i+1}}{|x|^2+1} = \frac{2y^{i+1}}{|y|^2+1}, \dots, \frac{|x|^2-1}{|x|^2+1} = \frac{|y|^2-1}{|y|^2+1}.$$

The last equality implies that $|x|^2 = |y|^2$ and substituting this in the previous equalities we get

$2x^1 = 2y^1, \dots, 2x^{i-1} = 2y^{i-1}, 2x^{i+1} = 2y^{i+1}, \dots, 2x^n = 2y^n$.

Moreover, using these and $|x|^2 = |y|^2$ we obtain $(x^i)^2 = (y^i)^2$.

$\Longrightarrow x^i = .y^i$ since $x^i, y^i \geq 0$.

Thus $x = y$ and hence φ is one to one on U^+.

Similarly φ is one to one on a neighborhood $U^-$of a point $p \in \mathbb{S}^n$ such that

$$U^- \subset \left\{x \in \overline{\mathbb{B}}^n : x^i \leq 0\right\}$$

Also φ^{-1} is also continuous.

Thus φ is a homeomorphism from a neighborhood U of p to an open subset $\varphi(U)$ of $\mathbb{H}^n$.

Moreover, if $x \in \partial\overline{\mathbb{B}}^n$, then $\varphi(x) \in \partial\mathbb{H}^n$, and hence if U is a neighborhood of x, then $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$.

Each point in $\mathbb{S}^{n-1} = \partial\overline{\mathbb{B}}^n$ is a boundary point.

Thus φ is a boundary chart.

Thus every point of $M = \overline{\mathbb{B}}^n$ is homeomorphic either to open subset of $\mathbb{R}^n$ or to $\mathbb{H}^n$.

Thus $\overline{\mathbb{B}}^n$ is n-dimensional manifold with boundary in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point.