Evaluation of a Class n-fold Integrals by Means of Hadamard Fractional Integration

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1 Introduction

In this note we will be evaluating a certain class of n-fold integrals over hypercubes via interpolation of the left Hadamard fractional integral operator. We won't be doing any fractional calculus other than a single interpolation theorem which may be used as a basis for fractional integration of the Hadamard type, note that this is not the more common Riemann nor Reiz types of fractional integral interpolation.

2 Main Body

2.1 Fractional Calculus

Fractional integrals are a generalization of n-fold iterated integrals to arbitrary order $\alpha \in \mathbb{C}$ (see theorem 2.2), there at least a few ways to do that, each giving rise to its own fractional calculus. [3, p. 1]

Definition 2.1. *The left Hadamard fractional integral operator will be denoted by* ${}_aI_x^\alpha$, *for* $0 < a < x < \infty, \Re[\alpha] > 0$ *is defined as*

$$ {}_a^H I_x^\alpha g(x) = \tfrac{1}{\Gamma(\alpha)} \int_a^x \log^{\alpha-1}\left(\tfrac{x}{t}\right) g(t) \tfrac{dt}{t} $$

assuming the integral is convergent and where log *denotes the natural logarithm,* Γ *is the usual gamma function, and* $\Re$ *is the real part.*

[3, p. 2]

Theorem 2.2. *Interpolation of this* n*-fold integral by the left Hadamard fractional integral operator.*

$$ \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n) \tfrac{dx_n \ldots dx_1}{x_n \cdots x_1} = {}_a^H I_x^n f(x) = \tfrac{1}{(n-1)!} \int_a^x \log^{n-1}\left(\tfrac{x}{t}\right) f(t)\, dt $$

Proof. The proof is by induction on n: (i) base case of $n = 1$ is obvious. (ii) Let $P(n)$ be the statement of theorem 2.2. Assume that $P(n)$ holds for some fixed positive integer n. Then,

$$\begin{aligned}
P(n+1) &= \int_a^x \int_a^{x_1} \cdots \int_a^{x_n} f\left(x_{n+1}\right) \tfrac{dx_{n+1}\dots dx_1}{x_{n+1}\cdots x_1} \\
&= \int_a^x \left[\int_a^{x_1} \cdots \int_a^{x_n} f\left(x_{n+1}\right) \tfrac{dx_{n+1}\dots dx_2}{x_{n+1}\cdots x_2}\right] \tfrac{dx_1}{x_1} \\
&= \int_a^x \left[\tfrac{1}{(n-1)!} \int_a^{x_1} \log^{n-1}\left(\tfrac{x_1}{t}\right) f\left(t\right) \tfrac{dt}{t}\right] \tfrac{dx_1}{x_1} \\
&= \tfrac{1}{(n-1)!} \int_a^x \int_t^x \log^{n-1}\left(\tfrac{x_1}{t}\right) f\left(t\right) \tfrac{dx_1 dt}{x_1 t} \\
&= \tfrac{1}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \left(\begin{array}{c} n-1 \\ k \end{array}\right) \int_a^x \int_t^x \log^k\left(x_1\right) \log^{n-k-1}\left(t\right) f\left(t\right) \tfrac{dx_1 dt}{x_1 t} \\
&= \tfrac{1}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \left(\begin{array}{c} n-1 \\ k \end{array}\right) \int_a^x \log^{n-k-1}\left(t\right) f\left(t\right) \tfrac{1}{t} \left[\int_t^x \log^k\left(x_1\right) \tfrac{dx_1}{x_1}\right] dt \\
&= \tfrac{1}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \left(\begin{array}{c} n-1 \\ k \end{array}\right) \tfrac{1}{k+1} \int_a^x \log^{n-k-1}\left(t\right) f\left(t\right) \tfrac{1}{t} \left(\log^{k+1} x - \log^{k+1} t\right) dt \\
&= \tfrac{1}{n!} \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right) \int_a^x \log^{n-k}\left(t\right) \log^k\left(x\right) f\left(t\right) \tfrac{dt}{t} \\
&= \tfrac{1}{n!} \int_a^x \log^n\left(\tfrac{x}{t}\right) f\left(t\right) \tfrac{dt}{t}
\end{aligned}$$

and the proof is complete. ■

The following non-standard definition we will adopt throughout the rest of this note because it is easier to work with than the standard definition 2.1 for our purposes.

Definition 2.3. *The modified left Hadamard fractional operator For* $0 < a < x < \infty, \Re[\alpha] > 0$

$$ {}_a^{eH}I_x^\alpha g\left(x\right) = \tfrac{1}{\Gamma(\alpha)} \int_0^{\log\left(\frac{x}{a}\right)} u^{\alpha-1} g\left(xe^{-u}\right) du$$

where we have substituted $u = \log\left(\frac{x}{t}\right)$ *in the integral of definition 2.1 and* $u^{\alpha-1}$ *is taken as it's principle value.*

2.2 Evaluations of n-fold integrals

Here we reduce the problem of evaluating certain n-fold integrals to that of solving a single fractional integral (just think of these as integrals transforms for our purposes).

Theorem 2.4. *Analytic continuation of certain* n*-fold integrals over unit hypercubes. Let* z *and* α *be a complex-valued parameters, let* t *denote a real variable, let* n *be a positive integer, and for fixed* $z = z_0$ *let* $f(z_0, t)$ *be a continuous function of* t *on* $[0, 1]$*. Then for suitable functions* $f(z, t)$ *(for which the integral converges) define*

$$F_n(z) := \int_0^1 \int_0^1 \cdots \int_0^1 f\left(z, \prod_{k=1}^{n} \lambda_k\right) d\lambda$$

where $d\lambda := d\lambda_n \dots d\lambda_1$ *(and likewise for other dummy variables as well). Then*

$$G(z,\alpha) := \tfrac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} f(z, e^{-u})\, du \text{ and } G(z,n) = F_n(z)$$

is the Hadamard fractional integral of order α *which is the analytic continuation of* $F_n(z)$ *from integer* n *to complex-valued* α *restricted to values for which the integral converges.*

Proof. [2] We will use the change of variables $y_k = \prod_{i=1}^{k} \lambda_i, k = 1, 2, \dots, n$ on the integral $F_n(z)$ to formulate an integral that represents the function for complex values of the argument via theorem 2.2 .

Note that the for given change of variables we have $\lambda_1 = y_1, \lambda_k = \frac{y_k}{y_{k-1}}, k = 2, 3, \dots, n$, hence

$$\frac{\partial \lambda_i}{\partial y_j} = \begin{cases} 1, & i = j = 1 \\ \frac{1}{y_{i-1}}, & i = j \neq 1 \\ -\frac{y_i}{y_{i-1}^2}, & i = j - 1 \\ 0, & \text{otherwise} \end{cases}$$

hence the Jacobian determinant is the product along the diagonal, $\left|\frac{\partial(\lambda_1,\dots,\lambda_n)}{\partial(y_1,\dots,y_n)}\right| = \frac{dy_n \dots dy_1}{y_{n-1}\cdots y_1}$. Notice that this change of variables maps the unit hypercube $[0,1]^n$ to the simplex

$$\{\vec{y} \in \mathbb{R}^n | 0 \leq y_1 \leq 1, 0 \leq y_i \leq y_{i-1}, \text{ for } i = 2, 3, \dots, n\}.$$

We replace the upper bound of y_1 with x so that

$$\begin{aligned} F_n(z) &= \lim_{a \to 0^+} \lim_{x \to 1^-} \int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} y_n f(z, y_n) \tfrac{dy_n \dots dy_1}{y_n \cdots y_1} \\ &= \lim_{a \to 0^+} \lim_{x \to 1^-} {}_a^{eH}I_x^n \left(x f(z,x)\right) \\ &= \tfrac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-t} f\left(z, e^{-t}\right) \end{aligned}$$

by theorem 2.2 which we analytically continue to

$$G(z,\alpha) = \lim_{a \to 0^+} \lim_{x \to 1^-} {}_a^{eH}I_x^\alpha \left[x f(z,x)\right] = \tfrac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} f(z, e^{-u})\, du$$

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The next few computer and table-assisted examples will illustrate the use of theorem 2.4.

Example 2.5. *[1, p. 193] Let* $G(z,\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} \cdot \frac{e^t}{(1+t)^z} dt$. *We see that this is a beta integral upon canceling* $e^{-t}e^t = 1$ *giving the value* $G(z,\alpha) = \frac{\Gamma(z-\alpha)}{\Gamma(z)}$. *We then determine what* $f(z,t)$ *is by comparing the integrand of the integral defining* $G(z,\alpha)$ *in this example to the corresponding integrand in theorem 2.4, to see that* $f(z, e^{-t}) = \frac{e^t}{(1+t)^z}$ *which implies that* $f(z,t) = \frac{t^{-1}}{(1-\log(t))^z}$ *and hence the evaluation of the n-fold integral of theorem 2.4 is*

$$F_n(z) = \int_0^1 \int_0^1 \cdots \int_0^1 \left(\prod_{k=1}^n \lambda_k\right)^{-1} \left(1 - \log \prod_{k=1}^n \lambda_k\right)^{-z} d\lambda = \tfrac{\Gamma(z-n)}{\Gamma(z)} = G(z,n).$$

Note that other integrals may be deduced from this by differentiation under the integral sign w.r.t. z, such as

$$F_n'(z) = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{-\log\left(1-\log\prod_{j=1}^n \lambda_j\right)}{\left(\prod_{m=1}^n \lambda_m\right)\left(1-\log\prod_{k=1}^n \lambda_k\right)^z} \, d\lambda = \frac{\Gamma(z-n)\left(\psi^{(0)}(z-n)-\psi^{(0)}(z)\right)}{\Gamma(z)}$$

where $\psi^{(m)}$ is the m^{th} derivative of the diagamma function.

Example 2.6. *[4] Let $G(z,\alpha,y) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\cdot\frac{e^{-(y-1)t}}{1-ze^{-t}}dt$. Wolfram.functions.com gives the value $G(z,\alpha,y) = \sum_{k=0}^\infty \frac{z^k}{(k+y)^\alpha} = \Phi(z,\alpha,y)$ where Φ is the Lerch Transcendent. We see that $f(z,t,y) = \frac{t^{y-1}}{1-zt}$ and hence the evaluation we seek is*

$$F_n(z,y) = \int_0^1 \int_0^1 \cdots \int_0^1 \left(1 - z\prod_{k=1}^n \lambda_k\right)^{-1} \prod_{j=1}^n \lambda_j^{y-1} \, d\lambda = \sum_{k=0}^\infty \frac{z^k}{(k+y)^n} = \Phi(z,n,y).$$

More integrals maybe calculated by differentiation under the integral sign w.r.t. y,

$$\begin{aligned}\frac{\partial F_n}{\partial y}(z,y) &= \int_0^1 \int_0^1 \cdots \int_0^1 \left(1 - z\prod_{k=1}^n \lambda_k\right)^{-1} \prod_{j=1}^n \lambda_j^{y-1} \cdot \log\prod_{\ell=1}^n \lambda_\ell d\lambda \\ &= -n\sum_{k=0}^\infty \frac{z^k}{(k+y)^{n+1}} = -n\Phi(z,n+1,y)\end{aligned}$$

Differentiating m times w.r.t. y, we get

$$\begin{aligned}\frac{\partial^m F_n}{\partial y^m}(z,y) &= \int_0^1 \int_0^1 \cdots \int_0^1 \left(1 - z\prod_{k=1}^n \lambda_k\right)^{-1} \prod_{j=1}^n \lambda_j^{y-1} \cdot \log^m\prod_{\ell=1}^n \lambda_\ell d\lambda \\ &= (-1)^m\frac{(n+m-1)!}{(n-1)!}\sum_{k=0}^\infty \frac{z^k}{(k+y)^{n+m}} = (-1)^m\frac{(n+m-1)!}{(n-1)!}\Phi(z,n+m,y)\end{aligned}$$

Acknowledgements

What does a man have that God has not given him? Nothing.
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References

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