

Relativistic Doppler Shift

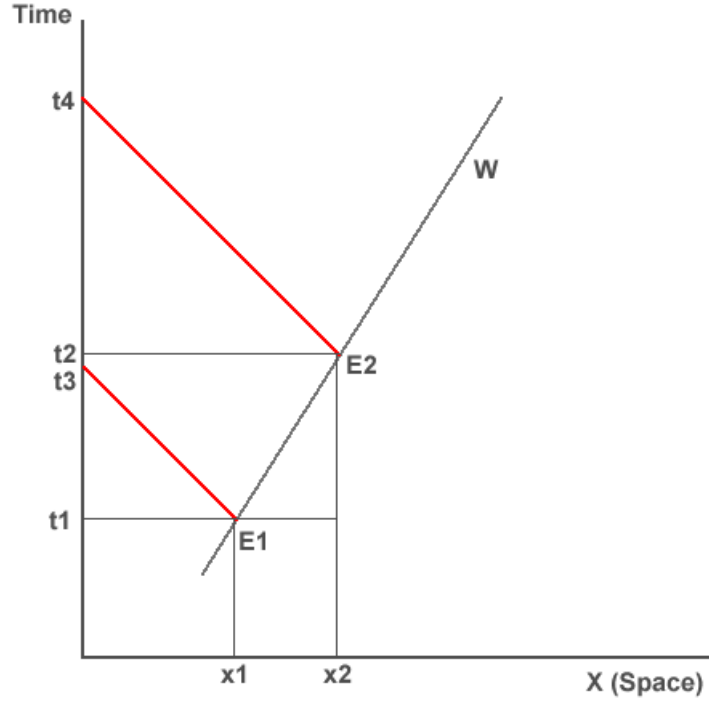
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The relativistic Doppler shift may be derived geometrically as follows.

1. Receding Worldlines

Figure 1: Time Measurement



The diagram shows an inertial time-like worldline W from which (at events E1 and E2) light pulses are sent to a receiver at rest in the diagram coordinates. The light pulses are received at times t_3 and t_4 by the receiver at $x = 0$.

The purpose of this exercise is to find the ratio of the proper times on W and the receiver R between the sending and receiving of the light pulses respectively. These intervals are $\tau_R = t_4 - t_3$ and the proper interval τ_E along W between E1 and E2.

Dealing with the latter we have

$$\tau_E^2 = (t_2 - t_1)^2 - (x_2 - x_1)^2 \quad (1)$$

$$= (t_2 - t_1)^2 - (v t_2 - v t_1)^2 \quad (2)$$

$$= (t_2 - t_1)^2 - v^2(t_2 - t_1)^2 \quad (3)$$

$$= (t_2 - t_1)^2 / \gamma^2 \quad (4)$$

and so $\tau_E = (t_2 - t_1) / \gamma$.

To find the corresponding interval $\tau_R = t_4 - t_3$ we use $t_3 = t_1 + x_1$ and $t_4 = t_2 + x_2$.

$$\tau_R = t_4 - t_3 = t_2 + x_2 - (t_1 + x_1) \quad (5)$$

$$= (t_2 - t_1) + (x_2 - x_1) = (t_2 - t_1) + v(t_2 - t_1) \quad (6)$$

$$= (t_2 - t_1)(1 + v) \quad (7)$$

The ratio we want is $\tau_R/\tau_E = \gamma(1 + v) = \sqrt{(1 + v)/(1 - v)}$

This result is the ratio of two invariants and so is itself invariant. It is independent of the frame in which the calculation is done and is therefore valid for any two separating senders and receivers with constant relative velocity v .

2. Approaching Worldlines

Equations (1)-(6) are independent of the sign of the velocity v and so still stand if the sign of v is reversed. However in equation (5) we must use $x_2 = x_1 - v(t_2 - t_1)$ instead of $x_2 = x_1 + v(t_2 - t_1) = vt_2$. With this substitution we find $\tau_R/\tau_E = \gamma(1 - v) = \sqrt{(1 - v)/(1 + v)}$